Phase transition in globally coupled Rössler oscillators

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(Received 30 November 1999)

We study a population of identical Rössler oscillators with global coupling. When the coupling constant is increased, an order-disorder-type phase transition occurs. Partial phase synchronization occurs in the ordered phase, although the amplitude of the oscillation is randomly distributed. We analyze the phase transition with a self-consistent method.

PACS number(s): 05.45.Xt, 64.60.Cn

Coupled oscillator models have been intensively studied as typical nonlinear-nonequilibrium systems [1–3]. Such coupled oscillator models are used for the description of Josephson-junction arrays or biological rhythms. Collective oscillatory motion appears in a population of oscillators with different natural frequencies or in a population of identical oscillators under external noises, when the mutual coupling is increased. A mean-field theory can be applied for the globally coupled oscillator systems. On the other hand, we have found a phase transition in a coupled map lattice where each dynamical element exhibits deterministic chaos [4,5]. We will show a macroscopic transition like a phase transition in a large population of identical Rössler oscillators [6].

The model equation is written as

$$\frac{dx_i}{dt} = -y_i - z_i + \frac{d}{N}\sum_{j=1}^{N}(x_j - x_i),$$

$$\frac{dy_i}{dt} = x_i + a y_i, \tag{1}$$

$$\frac{dz_i}{dt} = b + x_i z_i - c z_i,$$

where a, b, and c are parameters of the Rössler equation, d is the coupling constant, and N is the total number of oscillators. For parameters such as $a=0.15$, $b=0.2$, and $c=10$, the power spectrum of $x(t)$ manifests itself in sharp peaks, the phase of each oscillator is well defined, and the phase synchronization occurs easily. The phase synchronization is a phenomenon in which the amplitudes of chaotic oscillators are not synchronized but the phases are synchronized [7–9]. As a is increased for $b=0.2$ and $c=10$, the Rössler attractor becomes a funnel-like attractor and the projection of the attractor into the x-y plane does not have a hole region for $a>0.18$. For parameters such as $a=0.195$, $b=0.2$, and $c=10$, the peaks of the power spectrum of $x(t)$ are broad, and the oscillation is noisier. We will show numerical results for parameters $a=0.195$, $b=0.2$, and $c=10$. Figure 1 displays a time sequence of the average of x_i; i.e., $\langle x(t)\rangle=(1/N)\Sigma_{j=1}^{N}x_i(t)$ for $N=2000$. The numerical simuation was performed with the Runge-Kutta method. At $d=0.01$, the time sequence $\langle x(t)\rangle$ seems to be rather random and the amplitude of the averaged motion is small. However, the averaged motion $\langle x(t)\rangle$ exhibits a sinusoidal motion with a definite amplitude and a frequency at $d=0.017$. Each oscillator exhibits more chaotic time evolution; however, the averaged motion is fairly regular.

Figure 2 displays a snapshot of $[x_i(t), y_i(t)]$ for $i=1,2,\ldots,N$ at $d=0.01$ and 0.017. At $d=0.01$, $\langle x(t)\rangle\sim 0$; therefore, each oscillator's motion is almost independent and the snapshot profile of (x_i, y_i) is randomly distributed in the whole region of the Rössler attractor. At $d=0.017$, the collective motion appears and each oscillator tends to be synchronized by the collective oscillation. This phenomenon is similar to the order-disorder phase transition in thermodynamic systems. A similar transition was also found by Pikovsky, Rosenblum, and Kurths [8]. They mainly discussed mutual synchronization in a population of oscillators with different natural frequencies. We discuss the order-disorder transition in a population of identical oscillators with a self-consistent method.

We assume that the averaged motion can be approximated by a sinusoidal wave. That is, $1/N\Sigma_{j=1}^{N}x_j(t)=\langle x(t)\rangle$ is assumed to be $X_0+X_1\sin(\omega t)$, where X_0 denotes a time-averaged value of $\langle x(t)\rangle$, and X_1, ω denote the amplitude

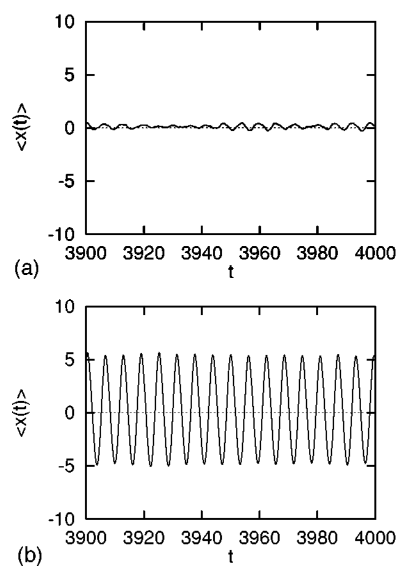

FIG. 1. Time sequences of the averaged motion $\langle x(t)\rangle=(1/N)\Sigma_{j=1}^{N}x_j(t)$ by Eq. (1) for $a=0.195, b=0.2, c=10$, and $N=2000$ at (a) $d=0.01$ and (b) 0.017.

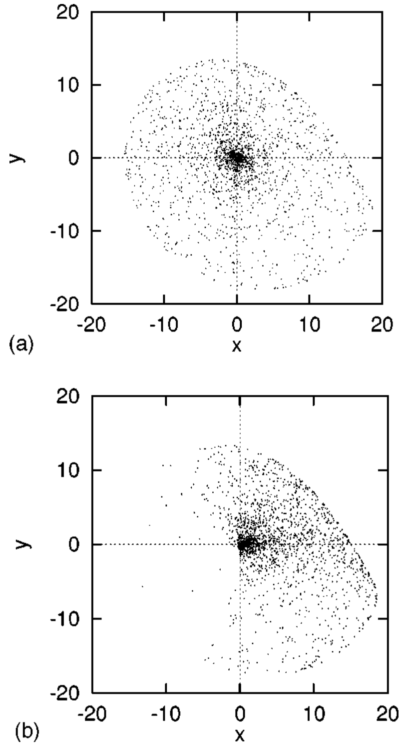

FIG. 2. Snapshot profiles of (x_i, y_j) at (a) $d=0.01$ and (b) 0.017.

and the frequency of the sinusoidal oscillation. Then the coupling term $d/N\Sigma_{j=1}^{N}(x_j - x_i)$ in the model equation (1) is reduced to $d[X_0 + X_1\sin(\omega t) - x_i]$. The equation for each oscillator is equivalent to the forced Rössler equation:

$$\frac{dx}{dt} = -y - z + d\{X_0 + f\sin(\omega t) - x\},$$

$$\frac{dy}{dt} = x + ay, \qquad (2)$$

$$\frac{dz}{dt} = b + xz - cz,$$

where the amplitude of the forcing f is equal to X_1. We have investigated Eq. (2) by changing f for certain fixed values of X_0 and ω. The time sequence of $x(t)$ is chaotic; however, the motion tends to be synchronized to the external periodic force. To measure the degree of the synchonization, we have calculated the amplitude of the sinusoidal component with frequency ω in the chaotic time sequence $x(t)$ by

$$X = (2/T)\int_{T_0}^{T_0+T} x(t)\sin(\omega t)dt.$$

The quantity X does not depend on an initial value of $x(t)$ or T_0 if T is sufficiently large. Figure 3 displays the relation of X and f for the parameters (a) $d=0.01, X_0=0.123, \omega=1.012$ and (b) $d=0.017, X_0=0.143, \omega=1.016$. The time interval $T=40\,000$ is used. The parameters X_0 and ω are numerically estimated values from the time sequence $\langle x(t)\rangle$ by the direct numerical simulation of Eq. (1). The amplitude X_1 of the averaged motion in Eq. (1) is written as

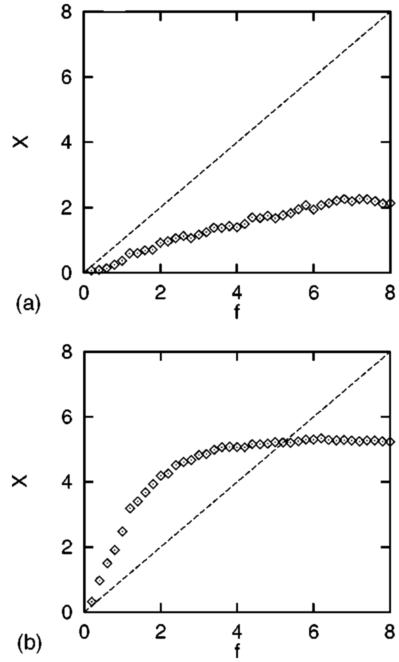

FIG. 3. Fourier amplitude X for the frequency ω as a function of f in the forced Rössler equation (2) at (a) $d=0.01$ and (b) 0.017.

$$X_1 = (2/T)\int_{T_0}^{T_0+T} \langle x(t)\rangle \sin(\omega t)dt$$

$$= (2/T)\int_{T_0}^{T_0+T} (1/N)\sum_{j=1}^{N} x_j(t)\sin(\omega t)dt$$

$$= 2/(NT)\sum_{j=1}^{N} \int_{T_0}^{T_0+T} x_j(t)\sin(\omega t)dt.$$

Each oscillator j obeys the same equation (2) independently and the quantity $(2/T)\int_{T_0}^{T_0+T} x_j(t)\sin(\omega t)dt$ takes the same value X for every j. The averaged value X_1 is therefore equal to X for the forced Rössler equation (2); that is, the amplitude X_1 of the averaged motion is equal to the temporal average X of the degree of the phase synchronization to the external periodic force. X_1 is also equal to f. The condition $X = X_1 = f$ represents the self-consistent condition that the averaged motion $\langle x(t)\rangle = X_0 + X\sin(\omega t)$ plays the role of the external force to each oscillator. The intersection of the $X=X(f)$ curve and $X=f$ gives a self-consistent solution. The intersection is $X=0$ for $d=0.01$. It implies that collective motion cannot occur for the parameter. The self-consistent solutions are $X=0$ and 5.21 for $d=0.017$. The solution $X=0$ may be unstable and the nontrivial solution $X=5.21$ is realized, which implies the appearance of the collective oscillation. On the other hand, we have calculated the amplitude of the collective oscillation in the time evolution of Eq. (1) by

$$A = \sqrt{(1/T)\int_{T_0}^{T_0+T} [\langle x(t)\rangle - X_0]^2 dt}.$$

If the averaged motion is expressed as $\langle x(t)\rangle = X_0 + X_1\sin(\omega t)$, A is equal to $A = X_1/\sqrt{2} = X/\sqrt{2}$. Figure 4 dis-

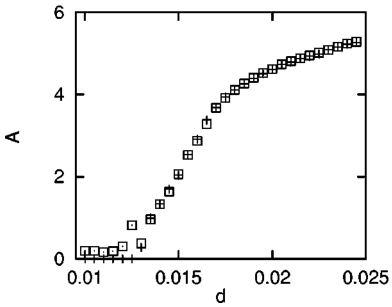

FIG. 4. Root-mean-square amplitude A of the collective oscillation as a function of the coupling constant. The square marks denote the values from the direct numerical simulation of Eq. (1) and the $+$ marks denote the values obtained by the self-consistent method shown in Fig. 3.

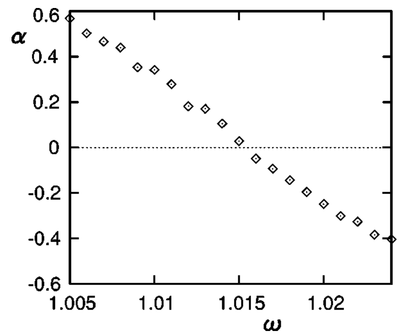

FIG. 5. Phase shift α between the time sequence $x(t)$ by the forced Rössler equation (2) and $\sin(\omega t)$ as a function of ω.

plays the numerically obtained values of A as a function of the coupling constant d. The points marked by squares denote A by the direct numerical simulation of Eq. (1) and the points marked by $+$ denote $A=X/\sqrt{2}$ obtained by the self-consistent method shown in Fig. 3. The values of X_0 and ω are assumed by $X_0=0.0968+2.63d$ and $\omega=1.005+0.654d$, which are linear fittings of X_0 and ω obtained from the averaged motion $\langle x(t)\rangle$ in the time evolution by Eq. (1) for several d. The self-consistent solution is a good approximation.

We have used a certain value of the frequency ω of the averaged motion to calculate Eq. (2). The frequency ω should also be obtained with a self-consistent method. If the frequency ω is changed in Eq. (2), $x(t)$ tends to be synchronized to the periodic force with the frequency ω. However, there may be a phase shift between the external force $f\sin(\omega t)$ and $x(t)$. To measure the phase shift, we have calculated $X=(2/T)\int_{T_0}^{T_0+T}x(t)\sin(\omega t)dt$ and $Y=(2/T)\int_{T_0}^{T_0+T}x(t)\cos(\omega t)dt$. The time-averaged phase shift is evaluated by

$$\alpha=\arctan(Y/X).$$

Figure 5 displays the relation of α and ω for $d=0.017$, $X_0=0.143$, and $f=5.22$, which are the values estimated from the direct numerical simulation of Eq. (1). The phase shift decreases as ω is increased. This is because $x(t)$ tends to lag behind the external force $\sin(\omega t)$, as the frequency ω is faster. The averaged motion of $x(t)$ should be equal to the external force by the self-consistent condition. It implies that the phase shift should be zero. The phase shift $\alpha\sim 0$ for $\omega\sim 1.0152$. It is close to the numerical value $\omega\sim 1.016$ by the direct simulation of Eq. (1) at $d=0.017$.

To summarize, we have analyzed an order-disorder-type phase transition in globally coupled Rössler oscillators with a self-consistent method. In the disordered phase, each oscillator's motion is nearly independent. Some phase synchronization occurs and collective oscillation appears in the ordered phase. The collective motion is assumed to be a simple sinusoidal oscillation, and the amplitude and the frequency have been numerically obtained by a self-consistent method from the chaotic motion of the forced Rössler equation. The self-consistent method gives a good intuitive interpretation for the appearance of the collective motion.

The averaged values of chaotic motion are generally not smooth functions of the parameters. For example, there may exist many window structures of periodic solutions in any parameter ranges. However, roughly speaking, the dynamical transition in a globally coupled chaotic oscillator is interpreted to be an analogue of the order-disorder-type phase transition in thermodynamic systems.

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